Stability of Fibonacci Functional Equation

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Abstract. In this paper, we solve the Fibonacci functional equation, \( f(x) = f(x-1) + f(x-2) \) and discuss its generalized Hyers-Ulam-Rassias stability in Banach spaces and stability in Fuzzy normed space.

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Introduction.

A question in the theory of functional equations is the following "When is it true that a function which approximately satisfies a functional equation must be close to an exact solution?" If the problem accepts a solution, we say that the equation is stable.

In 1940, S.M. Ulam [8] gave a wide-ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of important unsolved problems. Among those was the following question concerning the stability of homomorphism:

Let \( (G_1, *) \) be a group and \( (G_2, \circ, d) \) be a metric group with the metric \( d \). Given \( \epsilon > 0 \), does there exists a \( \delta_\epsilon > 0 \) such that if a mapping \( h: G_1 \rightarrow G_2 \) satisfies the inequality \( d(h(x*y),h(x) \circ h(y)) < \delta_\epsilon \) for all \( x, y \in G_1 \), then there is a mapping \( H: G_1 \rightarrow G_2 \) such that for each \( x, y \in G_1 \), \( H(x*y) = H(x) \circ H(y) \) and \( d(h(x),H(x)) < \epsilon \)?

In the next year, D. H. Hyers [3], gave answer to the above question for additive groups under the assumption that groups are Banach spaces. In 1978, T. M. Rassias [7] proved a generalization of Hyers' theorem for additive mapping as a special case in the form of following result.

Suppose that \( E \) and \( F \) are real normed spaces with \( F \) a complete normed space, \( f: E \rightarrow F \) is a mapping that for each fixed \( x \in E \) the mapping \( t \mapsto f(tx) \) is continuous on \( R \), and let there exist \( \epsilon \geq 0 \) and \( p \in [0,1) \) s.t

\[
\|f(x) - f(x-1) - f(x-2)\| \leq \epsilon
\]

for all \( x \in R \) and for some \( \epsilon > 0 \), Then there exists a unique linear mapping \( T: E \rightarrow F \) s.t

\[
\|f(x) - T(x)\| \leq \epsilon \frac{\|x\|^p}{(1 - 2^{p-1})}, x \in E.
\]

In this paper we discuss the stability of Fibonacci functional equation

\[
f(x) = f(x-1) + f(x-2). \tag{1}
\]

A function \( f: R \rightarrow X \) will be called a Fibonacci functional equation if it satisfies (1), for all \( x \in R \), where \( X \) is a real vector space. By \( \alpha \) and \( \beta \) we denote the positive and negative roots respectively of the quadratic equation \( x^2 - x - 1 = 0 \), i.e., \( \alpha = \frac{1+\sqrt{5}}{2} \) and \( \beta = \frac{1-\sqrt{5}}{2} \) for any \( x \in R \). M. M. Parizi and M. E. Gordji [11] proved the stability of Fibonacci functional equation in Modular functional spaces. S. M. Jung [10] also proved the stability of Fibonacci functional equation in real Banach space as following:

**Theorem1:** Let \( (X,|.|) \) be a real Banach space. If a function \( f: R \rightarrow X \) satisfies the inequality, \( \| f(x) - f(x-1) - f(x-2) \| \leq \epsilon \)

\[
(1.1)
\]

for all \( x \in R \) and for some \( \epsilon > 0 \), Then there exists a Fibonacci function \( F: R \rightarrow X \) such that \( \|f(x) - F(x)\| \leq \left(1 + \frac{2}{\sqrt{5}}\right) \epsilon \)

\[
(1.2)
\]

for all \( x \in R \).

Proof. We get from (1.1),

\[
||f(x) - \alpha f(x-1) - \beta [f(x-1) - \alpha f(x-2)]|| \leq \epsilon, \tag{1.3}
\]

For each \( x \in R \). If we replace \( x \) by \( x-k \) in (1.3), then we have,

\[
||f(x-k) - \alpha f(x-k-1) - \beta [f(x-k-1) - \alpha f(x-k-2)]|| \leq \epsilon
\]

And

\[
\|\beta^k [f(x-k) - \alpha f(x-k-1)] - \beta^{k+1} [f(x-k-1) - \alpha f(x-k-2)]\| \leq \beta^k \epsilon \tag{1.4}
\]
Thus, we have,
\[
\|f(x) - \alpha f(x-1) - \beta^n [f(x-n) - \alpha f(x-n-1)]\| \leq \sum_{k=0}^{n-1} \|\beta^k [f(x-k) - \alpha f(x-k-1)] - \beta^{k+1} [f(x-k-1) - \alpha f(x-k-2)]\| \leq \sum_{k=0}^{n-1} |\beta|^k \epsilon
\]
(1.5)
From (1.4), we get \(\{\beta^n [f(x-n) - \alpha f(x-n-1)]\}\) is a Cauchy sequence. Therefore, we can define a function \(F_1 : \mathbb{R} \to X\) by
\[
F_1 = \lim_{n \to \infty} \beta^n \alpha^n [f(x-n) - \alpha f(x-n-1)] \text{, since } X \text{ is complete so } F_1 \text{ is in } X.
\]
We obtain that
\[
F_1(x-1) + F_1(x-2) = \beta^{-1} \lim_{n \to \infty} \beta^n [f(x-n-1) - \alpha f(x-n)] + \beta^{-2} \lim_{n \to \infty} \beta^{n+2} [f(x-(n+2)) - \alpha f(x-(n+2)-1)]
\]
\[
= \beta^{-1} F_1(x) + \beta^{-2} F_1(x) = F_1(x),
\]
For all \(x \in \mathbb{R}\). Hence \(F_1\) is a Fibonacci function. If \(n\) goes to infinity, then (1.5) implies
\[
\|f(x) - \alpha f(x-1) - F_1(x)\| \leq \frac{\sqrt{\epsilon}}{\sqrt{2}} \epsilon
\]
(1.6)
For every \(x \in \mathbb{R}\).

From (1.1)
\[
\|f(x) - \beta f(x-1) - \alpha [f(x-1) - \beta f(x-2)]\| \leq \epsilon
\]
(1.7)
For each \(x \in \mathbb{R}\). If we replace \(x\) by \(x+k\) in (1.7), then we have,
\[
\|f(x+k) - \beta f(x+k-1) - \alpha [f(x+k-1) - \beta f(x+k-2)]\| \leq \epsilon
\]
And
\[
\|\alpha^n [f(x+k) - \beta f(x+k-1)] - \alpha^{n+1} [f(x+k-1) - \beta f(x+k-2)]\| \leq \alpha^n \epsilon
\]
(1.8)
Thus, we have,
\[
\|\alpha^n [f(x+n) - \beta f(x+n-1)] - f(x) - \beta f(x-1)]\| \leq \sum_{k=0}^{n-1} \|\alpha^{-k} [f(x+k) - \beta f(x+k-1)] - \alpha^{-k-1} [f(x+k-1) - \beta f(x+k-2)]\| \leq \sum_{k=0}^{n-1} |\alpha|^{-k} \epsilon
\]
(1.9)
From (1.8), we get \(\{\alpha^n [f(x+n) - \beta f(x+n-1)]\}\) is a Cauchy sequence. Therefore, we can define a function \(F_2 : \mathbb{R} \to X\) by
\[
F_2 = \lim_{n \to \infty} \alpha^{-n} [f(x+n) - \beta f(x+n-1)] \text{, since } X \text{ is complete so } F_2 \text{ is in } X.
\]
We obtain that
\[
F_2(x-1) + F_2(x-2) = \alpha^{-1} \lim_{n \to \infty} \alpha^{-n+1} [f(x+n-1) - \beta f(x+n-1)] + \alpha^{-2} \lim_{n \to \infty} \alpha^{-n+2} [f(x+n-2) - \beta f(x+n-2-1)]
\]
\[
= \alpha^{-1} F_2(x) + \alpha^{-2} F_2(x) = F_2(x),
\]
For all \(x \in \mathbb{R}\). Hence \(F_2\) is a Fibonacci function. If \(n\) goes to infinity, then (1.9) implies
\[
\|F_2(x) - f(x) + \beta f(x-1)\| \leq \frac{\sqrt{\epsilon}}{\sqrt{2}} \epsilon
\]
(1.10)
For every \(x \in \mathbb{R}\).
From (1.6) and (1.10), we have
\[
||f(x)-\frac{\beta}{\beta-a}F_1(x)-\frac{\alpha}{\beta-a}F_2||=\frac{1}{|\beta-a|}||((\beta-\alpha)f(x)-[\beta F_1(x)-\alpha F_2(x)]||\leq \frac{1}{\alpha-\beta}||f(x)-\alpha F_2(x)||+\frac{1}{\alpha-\beta}||\beta F_1(x)|| \leq (1+\sqrt{5})\epsilon
\]
For all xєR. Now we set
\[
F(x)=\frac{\beta}{\beta-a}F_1(x)-\frac{\alpha}{\beta-a}F_2
\]
Clearly F(x) is the Fibonacci function.

Now we prove the stability of Fibonacci functional equation in fuzzy normed space.

**Theorem 2:** Let (X, N) and (Y, N') be fuzzy normed spaces. If f: R→X satisfies the inequality
\[
N(f(x)-f(x-1)-f(x-2), t) \geq N'(\phi(x), t) \tag{2.1}
\]
for all xєR, then there exists a Fibonacci function F: R→X such that
\[
N(f(x)-F(x), t) \geq N'(\phi(x), (1+\frac{2}{\sqrt{5}})t) \tag{2.2}
\]
\[
N((\beta^n[f(x-n)-\alpha f(x-(n-1))],[\beta^n F_1(x)-\alpha F_2(x)], \beta^n t) \geq N'(\phi(x), t) \tag{2.3}
\]
Thus, we have,
\[
N(f(x)-f(x-1)-[f(x-n)-\alpha f(x-(n-1))], \beta^n t) \geq \min(N(\beta^n f(x)-\alpha f(x-1)-[f(x-1)-\alpha f(x-2)], \beta^n t), k=0,1,\ldots, n-1) \geq N(\phi(x), \sum_{k=0}^{n-1} |\beta|^k t) \tag{2.4}
\]
From (2.4), we get \(\beta^n[f(x-n)-\alpha f(x-(n-1))]\) is a Cauchy sequence. Therefore, we can define a function \(F_1: R→X\) by
\[
F_1=\lim_{n→∞} \beta^n[f(x-n)-\alpha f(x-(n-1))], \text{ since X is complete so } F_1 \text{ is in } X. \text{ We obtain that}
\[
F_1(x+1)+F_1(x-2)=\beta^{-1} \lim_{n→∞} \beta^{n+1}[f(x-n-1)-\alpha f(x-n)] +\beta^{-2} \lim_{n→∞} \beta^{n+2}[f(x-(n+2)) - \alpha f(x-(n+2) - 1)]
\]
\[
=\beta^{-1}F_1(x)+ \beta^{-2}F_1(x)=F_1(x),
\]
For all xєR. Hence \(F_1\) is a Fibonacci function. If n goes to infinity, then (2.4) implies
\[
N(f(x)-f(x-1)-F_1(x), t) \geq N'(\phi(x), (1+\frac{2}{\sqrt{5}})t) \tag{2.5}
\]
For every \( x \in \mathbb{R} \).

From (2.1)

\[ N(f(x) - \beta f(x-1)) \geq N'(\alpha f(x), t), \]

(2.6)

For each \( x \in \mathbb{R} \). If we replace \( x \) by \( x+k \) in (2.6), then we have,

\[ N(f(x+k) - \beta f(x+k-1)) \geq N'(\alpha f(x+k), t) \]

And

\[ N(\alpha^k [f(x+k) - \beta f(x+k-1)] - \alpha^{k+1} [f(x+k-1) - \beta f(x+k-2)], t) \geq N'(\alpha f(x+k), t) \quad \text{(2.7)} \]

Thus, we have,

\[ N(\alpha^{-n} [f(x+n) - \beta f(x+n-1)] - \alpha^{-1} \sum_{k=0}^{n-1} \alpha^{-k} t) \geq \]

\[ \min\{N(\alpha^{-n} [f(x+n-1) - \beta f(x+n-2)], t), N(\alpha f(x+n) - \beta f(x+n-1)], t) \}

(2.8)

From (2.7), we get \((\alpha^{-n} [f(x+n) - \beta f(x+n-1)])\) is a Cauchy sequence. Therefore, we can define a function \( F_2 : \mathbb{R} \to \mathbb{X} \) by

\[ F_2(x) = \lim_{n \to \infty} \alpha^{-n} [f(x+n) - \beta f(x+n-1)] \]

since \( \mathbb{X} \) is complete so \( F_2 \) is in \( \mathbb{X} \). We obtain that

\[ F_2(x) = \alpha^{-1} \lim_{n \to \infty} \alpha^{-n+1} [f(x+n) - \beta f(x+n-1)] \]

\[ + \alpha^{-2} \lim_{n \to \infty} \alpha^{-n+2} [f(x+n-1) - \beta f(x+n-2)] - \beta f(x+(n-1)) \]

\[ = \alpha F_2(x) + \alpha F_2(x) = F_2(x), \]

For all \( x \in \mathbb{R} \). Hence \( F_2 \) is a Fibonacci function. If \( n \) goes to infinity, then (2.8) implies

\[ N(F_2(x) + \beta f(x-1), t) \geq N'(\alpha f(x), \left(\sqrt{\frac{\alpha^2+1}{2}}\right)t) \]

(2.9)

For every \( x \in \mathbb{R} \).

From (2.5) and (2.9), we have

\[ N(f(x) - \beta f(x-1)) \geq N'(\alpha f(x), \left(1 + \frac{\alpha}{\sqrt{\frac{\alpha^2+1}{2}}}t) \]

(2.10)

For all \( x \in \mathbb{R} \). Now we set

\[ F(x) = \frac{\beta}{\beta - \alpha} F_1(x) - \frac{\alpha}{\beta - \alpha} F_2 \]

Clearly \( F(x) \) is the Fibonacci function.

References


